

q -CHAOS

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ABSTRACT. We consider the L_p norm estimates for homogeneous polynomials of q -gaussian variables ($-1 \leq q \leq 1$). When $-1 < q < 1$ the L_p estimates for $1 \leq p \leq 2$ are essentially the same as the free case ($q = 0$), whilst the L_p estimates for $2 \leq p \leq \infty$ show a strong q -dependence. Moreover, the extremal cases $q = \pm 1$ produce decisively different formulae.

1. INTRODUCTION

In classical probability theory and in analysis the Wiener chaos is well understood and fundamental. The orthogonal polynomials for the gaussian measure in \mathbb{R}^n are given by the Hermite polynomials. In this paper we are interested in norm estimates for homogeneous polynomials in q -gaussian variables. The family of q -gaussian variables, introduced by Bozejko and Speicher ([3]), is a natural noncommutative generalization of classical gaussian random variables. Here q ranges between $[-1, 1]$. The case $q = 1$ corresponds to the classical situation, $q = -1$ reflects the fermionic case, and $q = 0$ comes from Voiculescu's free probability theory. Our work is motivated by the beautiful results by Haagerup/Pisier ([7]) on the operator space structure of the space of homogeneous polynomials in the free group. Similar result have been obtained by Ricard/Xu ([21]) for free products. We want to show that norm estimates for polynomials of degree 2 and higher can detect the value q .

The space of polynomials of degree one is just the linear span of the (q -) gaussian variables and are, by now, very well understood through Khintchine type inequalities. The starting point of these result is Lust-Piquard's Khintchine inequality ([13]), later extended by Lust-Piquard/Pisier to the range $1 \leq p \leq 2$ ([14]). In the free case, Buchholz ([1]) provided very precise estimates. For general information on free chaos we refer to the work of [10], which stems from earlier work of Pisier/Parcet ([17]). For $p = \infty$, very nice estimates for polynomials can be found in [21], the L_p versions can be found in [10].

Analyzing the known norm estimates for the span of the generators of q -gaussian random variables, it turns out that the dependence on q disappears. Indeed, any suitable notion of gaussian random variables leads to similar expression for a fixed value of p . In this paper we provide formulas for homogeneous polynomials of higher degree and show that already for polynomials of degree 2 the results do depend on q , however the dependence is quite subtle.

Theorem 1.1. *Let $-1 < q < 1$ and $(g_{q,i})_{i=1}^m$ (resp. $(g_i)_{i=1}^m$) be q -gaussian (resp. free-gaussian) random variables defined with respect to some reference state ϕ_q (resp. ϕ) with density D_q (resp. D). Let $(x_{ij})_{i,j=1}^m$ be $L_p(\mathcal{M})$ valued coefficients for a von Neumann algebra \mathcal{M} and $1 \leq p \leq 2$. Then*

$$\left\| \sum_{i \neq j} x_{ij} \otimes D_q^{\frac{1}{2p}} g_{q,i} g_{q,j} D_q^{\frac{1}{2p}} \right\|_p \sim_{c_q} \left\| \sum_{i \neq j} x_{ij} \otimes D^{\frac{1}{2p}} g_i g_j D^{\frac{1}{2p}} \right\|_p.$$

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The good news is that the right hand side can be calculated (up to some universal constant) by a formula involving the best decomposition with respect to three norms (a generalization of the basic \mathcal{K} - functional spaces in interpolation). Such decomposition norms should be considered classical in the theory as in so far they appear already in Lust-Piquard/Pisier's operator valued Khintchine inequality for the range $1 \leq p \leq 2$. We refer to section 3 for a precise formulation. It seems that Theorem 1.1 has no dependence in q (except the constant). However, there are singularities for $q = 1$ and $q = -1$. In the special cases we have to use a decoupling technique. Indeed, it is well-known from Banach space theory (see e.g. the book of Ledoux and Talagrand ([15])) that

$$\mathbb{E} \left\| \sum_{i \neq j} x_{ij} \otimes \varepsilon_i \varepsilon_j \right\|_X \sim_c \mathbb{E} \left\| \sum_{i \neq j} x_{ij} \otimes \varepsilon_i \varepsilon'_j \right\|_X$$

holds for Banach space valued coefficients $x_{ij} \in X$. Here $\varepsilon_i, \varepsilon'_j$ are independent coordinate functions defined on $\{-1, 1\}^n \times \{-1, 1\}^n$ and \mathbb{E} is the corresponding expectation. Based on new decoupling techniques we prove the following.

Theorem 1.2. *Let $q = \pm 1$ and $(g_{q,i})_{i=1}^m$ be q -gaussian random variables defined with respect to some reference state ϕ_q with density D_q , and let $(g_i)_{i=1}^m$ and D be as in Theorem 1.1. Let $1 \leq p < \infty$ and $(x_{ij})_{i,j=1}^m$ be $L_p(\mathcal{M})$ valued coefficients for a von Neumann algebra \mathcal{M} .*

i) *Let $q = -1$ and $x_{ij} = -x_{ji}$. Then*

$$\left\| \sum_{i,j} x_{ij} \otimes D_{-1}^{\frac{1}{2p}} g_{-1,i} g_{-1,j} D_{-1}^{\frac{1}{2p}} \right\|_p \sim \left\| \sum_{i,j} x_{ij} \otimes D^{\frac{1}{2p}} g_i D^{\frac{1}{2p}} \otimes D^{\frac{1}{2p}} g_j D^{\frac{1}{2p}} \right\|_p.$$

ii) *Let $q = 1$ and $x_{ij} = x_{ji}$. Then*

$$\left\| \sum_{i,j} x_{ij} \otimes D_1^{\frac{1}{2p}} g_{1,i} g_{1,j} D_1^{\frac{1}{2p}} \right\|_p \sim \left\| \sum_{i,j} x_{ij} \otimes D^{\frac{1}{2p}} g_i D^{\frac{1}{2p}} \otimes D^{\frac{1}{2p}} g_j D^{\frac{1}{2p}} \right\|_p.$$

Here again the right hand side can be calculated using martingale inequalities for the linear terms. This leads to a four term maximum for $p \geq 2$ and a four term decomposition for $1 \leq p \leq 2$. The formulae are decisively different from the three term expression in Theorem 1.1. Let us refer to section 4 for the concrete expressions. Moreover, in section 3 we also extend this result to polynomials of arbitrary degree. The notion becomes rather involved and the estimates depend on the degree of the polynomial. In spirit our method is closely related to similar estimates for polynomials on the free group by Haagerup and Piser. However, in our approach the decoupling is derived from Speicher's central limit procedure combined with the ultraproduct technique from [9].

Finally, let us come back to the q -dependence for polynomials of degree 2. For $p \geq 2$ we can use duality arguments starting from Theorem 1.1. These estimates are based on the previous work of Nou ([16]). Indeed, a classical tool for studying q -gaussian variables is the Wick order. This order essentially implements the identification between the GNS-Hilbert space given by the vacuum state and the Fock space realization. In our situation the usual product $g_{q,i} g_{q,j}$ ($i \neq j$) coincides with the Wick product, however understanding $g_{q,i} g_{q,j}$ as a Wick product plays an essential role when $p \geq 2$.

Theorem 1.3. *Let $-1 < q < 1$, $(x_{ij})_{i,j=1}^m$ be $L_p(\mathcal{M})$ valued coefficients for a von Neumann algebra \mathcal{M} and $2 \leq p \leq \infty$. Let $(g_{q,i})_{i=1}^m$, D_q , $(g_i)_{i=1}^m$ and D be as in*

Theorem 1.1. Then

$$\left\| \sum_{i \neq j} x_{ij} \otimes D_q^{\frac{1}{2p}} g_{q,i} g_{q,j} D_q^{\frac{1}{2p}} \right\|_p \sim_{c_q} \left\| \sum_{i \neq j} (x_{ij} + q x_{ji}) \otimes D_q^{\frac{1}{2p}} g_i g_j D_q^{\frac{1}{2p}} \right\|_p.$$

Moreover, the span of polynomials of degree 2 is completely complemented in the corresponding L_p space with a constant depending on q .

The q -dependent term $x_{ij} + q x_{ji}$ above comes from the symmetrization operator P_2 on \mathcal{H}^2 for a Hilbert space \mathcal{H} defined by

$$P_2(f_1 \otimes f_2) = f_1 \otimes f_2 + q f_2 \otimes f_1$$

for any $f_1, f_2 \in \mathcal{H}$. See section 2.1 for the details. Again the result also holds for polynomials of higher degree, but we refer to the text for the explicit formulation.

The paper is organized as follows. In section 2 we present some preliminaries we need in sequel. That includes the Fock space realization of the (generalized) q -gaussian random variables for $-1 \leq q \leq 1$, Wick product and some modular theory for the case $-1 < q < 1$, and matrix models for the case $q = \pm 1$. At the end of the section 2 we collect some notations mainly concerned with complicated indices we will encounter. In section 3 we focus on the case $-1 < q < 1$. We first establish the free case using the result in [10] and obtain an appropriate interpolation scale. Using this interpolation scale and the approach of Nou ([16]) we can get the result for the general case $-1 < q < 1$. In the final section we consider the remaining cases $q = \pm 1$.

We assume that the reader is familiar with standard concepts in operator algebra ([23, 24]), operator space theory ([5, 19]), noncommutative L_p spaces ([11, 20]), and the related complex interpolation theory ([12, 25, 20]). For a von Neumann algebra \mathcal{M} we denote the noncommutative L_p ($1 \leq p \leq \infty$) space with respect to \mathcal{M} by $L_p(\mathcal{M})$. When $E \subseteq L_p(\mathcal{N})$ for another von Neumann algebra \mathcal{N} , the norm closure of the algebraic tensor product $L_p(\mathcal{M}) \otimes E$ in $L_p(\mathcal{M} \bar{\otimes} \mathcal{N})$ will be denoted by $L_p(\mathcal{M}; E)$. For a Hilbert space H we write $S_p(H) = L_p(B(H))$ and H_r and H_c imply the row and the column Hilbert space on H , respectively. When $H = \ell_2$ we simply write S_p , R and C , respectively. R_p and C_p imply the linear space of the first row and column of S_p , respectively.

2. PRELIMINARIES AND NOTATIONS

2.1. The Fock space realization. We start with the Fock space realization of q -commutation relations for $-1 \leq q \leq 1$. Let \mathcal{H} be an infinite dimensional separable complex Hilbert space equipped with an orthonormal basis $(e_{\pm k})_{k \geq 1}$. We denote by $\mathcal{F}_0(\mathcal{H})$ the associated free Fock space

$$\mathcal{F}_0(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n},$$

where Ω is a unit vector called vacuum. We consider the operator of symmetrization P_n on $\mathcal{H}^{\otimes n}$ defined by

$$P_0 \Omega = \Omega, \\ P_n(f_1 \otimes \cdots \otimes f_n) = \sum_{\pi \in S_n} q^{i(\pi)} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)},$$

where S_n denotes the symmetric group of permutations of n elements and

$$i(\pi) = \#\{(i, j) | 1 \leq i, j \leq n, \pi(i) > \pi(j)\}$$

is the number of inversions of $\pi \in S_n$.

Now we define the q -inner product $\langle \cdot, \cdot \rangle_q$ on $\mathcal{F}_0(\mathcal{H})$ by

$$\langle \xi, \eta \rangle_q = \delta_{n,m} \langle \xi, P_n \eta \rangle \quad \text{for } \xi \in \mathcal{H}^{\otimes n}, \eta \in \mathcal{H}^{\otimes m},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in \mathcal{H} . Since P_n 's are strictly positive for $-1 < q < 1$ ([3]), $\langle \cdot, \cdot \rangle_q$ is actually an inner product. In this case, the Hilbert space $\mathcal{F}_0(\mathcal{H})$ equipped with q -inner product is called q -Fock space on \mathcal{H} , and we denote it by $\mathcal{F}_q(\mathcal{H})$. When $q = \pm 1$, P_n 's are just positive, so we define $\mathcal{F}_q(\mathcal{H})$ as the quotient of $(\mathcal{F}_0(\mathcal{H}), \langle \cdot, \cdot \rangle_q)$ by the corresponding kernel.

For two sequences of strictly positive reals $\lambda = (\lambda_k)_{k \geq 1}$ and $\mu = (\mu_k)_{k \geq 1}$ we define q -(generalized) gaussian variables (or q -generalized circular variables) by

$$g_{q,k} = \lambda_k \ell_q(e_k) + \mu_k \ell_q^*(e_{-k}),$$

where $\ell_q(h)$ is the left creation operator by $h \in \mathcal{H}$ and $\ell_q^*(h)$ is the adjoint of $\ell_q(h)$. It is easy to check that $g_{q,k}$'s satisfy the q -commutation relations

$$g_{q,k}^* \cdot g_{q,j} - q \cdot g_{q,j} \cdot g_{q,k}^* = \delta_{kj}(\lambda_k^2 + \mu_k^2)I.$$

When $q = \pm 1$ we have additional relations

$$g_{q,k} \cdot g_{q,j} - q \cdot g_{q,j} \cdot g_{q,k} = 0$$

which implies that $g_{q,k}$'s are CAR and CCR sequences in the corresponding cases.

We focus on Γ_q ($-1 \leq q < 1$), the von Neumann algebra generated by $\{g_{q,k}\}_{k \geq 1}$. When $q = 1$ we define Γ_1 by the von Neumann algebra generated by $\{\exp(i \cdot g_{1,k})\}_{k \geq 1}$ since $g_{1,k}$'s are unbounded operators in this case.

There is a canonical way to translate the above picture into the framework of Shlyakhtenko and Hiai ([22, 6]). According to section 4 in [22] we can associate $g_{q,k}$ with an action

$$U_t^k = \begin{pmatrix} \cos \theta_k t & -\sin \theta_k t \\ \sin \theta_k t & \cos \theta_k t \end{pmatrix},$$

where $\theta_k = \log[(\frac{\mu_k}{\lambda_k})^2]$ on $H_k(\cong \mathbb{R}^2)$. Note that the basis $(\tilde{e}_k, \tilde{e}_{-k})$ on H_k for this matrix representation is given by

$$(2.1) \quad \begin{pmatrix} \tilde{e}_k \\ \tilde{e}_{-k} \end{pmatrix} = \frac{1}{\sqrt{\alpha_k + 1}} \begin{pmatrix} -i & \sqrt{\alpha_k} i \\ 1 & \sqrt{\alpha_k} \end{pmatrix} \begin{pmatrix} e_k \\ e_{-k} \end{pmatrix} = V \begin{pmatrix} e_k \\ e_{-k} \end{pmatrix},$$

where $\alpha_k = \lambda_k^{-2} \mu_k^2$.

Let $s(\tilde{e}_k)$ and $s(\tilde{e}_{-k})$ be semi-circular variables defined by

$$s(\tilde{e}_{\pm k}) = \frac{1}{2}(\ell(\tilde{e}_{\pm k}) + \ell^*(\tilde{e}_{\pm k})).$$

Then we have

$$(2.2) \quad g_{q,k} = \frac{\sqrt{\lambda_k^2 + \mu_k^2}}{2} (s(\tilde{e}_k) + i s(\tilde{e}_{-k})).$$

Since \mathcal{H} is the complexification of $H_{\mathbb{R}} = \bigoplus_{k \geq 1} H_k$, by setting

$$U_t = \bigoplus_{k \geq 1} U_t^k$$

we get that

$$\Gamma_q(H_{\mathbb{R}}, U_t) = \{s(e_{\pm k}) : k \geq 1\}'' = \Gamma_q.$$

In [22] we introduced another inner product $\langle \cdot, \cdot \rangle_U$ on \mathcal{H} defined by

$$\langle x, y \rangle_U = \langle 2A(1+A)^{-1}x, y \rangle,$$

where A is the operator satisfying

$$U_t = A^{it}.$$

Note that $A = \bigoplus_{k \geq 1} A_k$ with $U_t^k = A_k^{it}$, and by taking conjugate with respect to the basis change matrix V in (2.1) we get the matrix representation of A_k with respect to (e_k, e_{-k}) by

$$V^{-1}A_kV = \begin{pmatrix} \lambda_k^2 \mu_k^{-2} & 0 \\ 0 & \lambda_k^{-2} \mu_k^2 \end{pmatrix},$$

which implies that

$$(2.3) \quad Ae_k = \lambda_k^2 \mu_k^{-2} e_k$$

for any $k \geq 1$.

2.2. The Wick product and some modular theory for the case $-1 < q < 1$.

Let $-1 < q < 1$. Since it is well known that Ω is separating for Γ_q , for every $\xi \in \Gamma_q \Omega$ there exist a unique operator $W(\xi) \in \Gamma_q$ such that

$$W(\xi)\Omega = \xi.$$

This W is called the *Wick product*. There is a useful decomposition of the Wick product as follows.

$$(2.4) \quad W(\xi) = \sum_{k=0}^n U_k R_{n,k}^*(\xi)$$

for $\xi \in \mathcal{H}^{\otimes n}$, where $R_{n,k}$ is the operator on $\mathcal{H}^{\otimes n}$ given by

$$R_{n,k}(f_1 \otimes \cdots \otimes f_n) = \sum_{\pi \in S_n / S_{n-k} \times S_k} q^{i(\pi)} f_{\pi^{-1}(1)} \otimes \cdots \otimes f_{\pi^{-1}(n)}$$

and $U_k : \mathcal{H}_c^{\otimes n-k} \otimes \mathcal{H}_r^{\otimes k} \rightarrow B(\mathcal{F}_q(\mathcal{H}))$ is the operator defined by

$$U_k(e_1 \otimes \cdots \otimes e_n) = \ell(e_1) \cdots \ell(e_{n-k}) \ell^*(e_{n-k+1}) \cdots \ell^*(e_n).$$

In the above $S_n / S_{n-k} \times S_k$ means the representatives of the right cosets of $S_{n-k} \times S_k$ in S_n with minimal numbers of inversions. It is well known that

$$(2.5) \quad P_n = R_{n,k}(P_{n-k} \otimes P_k)$$

and by Corollary 1 in [16]

$$(2.6) \quad \|U_k : H_c^{\otimes n-k} \otimes_h H_r^{\otimes k} \rightarrow B(\mathcal{F}_q(\mathcal{H}))\|_{cb} \leq C_q,$$

where $H^{\otimes k}$ is the Hilbert space on $\mathcal{H}^{\otimes k}$ equipped with the q -inner product and

$$C_q = \prod_{n \geq 1} \frac{1}{1 - q^n}.$$

We can estimate the norm of P_n and its inverse. In [4] it is shown that

$$P_n \leq \frac{1}{1-q} I_1 \otimes P_{n-1},$$

where I_1 is the formal identity on H , thus by repeating the above operator inequality we have

$$(2.7) \quad \|P_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}\| \leq \left(\frac{1}{1-q} \right)^{n-1}.$$

Moreover, it is shown in [2] that $I_1 \otimes P_{n-1} \leq w(q)^{-1} P_n$ for some constant $w(q) > 0$, thus by repeating again we have

$$(2.8) \quad \|P_n^{-1} : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}\| \leq w(q)^{-n+1}.$$

From the definition it is clear that

$$(2.9) \quad P_n^{\frac{1}{2}} : H^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$$

is an isometry.

We close this section with some modular theory for Γ_q and ϕ_q , the vacuum state defined by $\phi_q(\cdot) = \langle \Omega \cdot, \Omega \rangle_q$. It is well known that the modular group σ_t with respect to ϕ_q satisfies the following.

$$\sigma_t(g_{q,k}) = (\lambda_k^{-1} \mu_k)^{2it} g_{q,k}.$$

Thus, $g_{q,k}$ is an analytic element satisfying

$$(2.10) \quad D_q^{\frac{1}{2p}} g_{q,k} = (\lambda_k^{-1} \mu_k)^{\frac{1}{p}} g_{q,k} D_q^{\frac{1}{2p}}.$$

Recall that the anti-linear map S is the closure of the operator given by

$$S(x\Omega) = x^*\Omega$$

for all $x \in \Gamma_q$. Then S can be written as $S = J\Delta^{\frac{1}{2}}$, where J is the modular conjugation and Δ is the modular operator. By [6] we know that Δ is the closure of the operator $\bigoplus_{n \geq 0} (A^{-1})^{\otimes n}$, which implies

$$(2.11) \quad S|_{H^{\otimes n}} = J(A^{-\frac{1}{2}})^{\otimes n}.$$

2.3. Matrix models for CAR and CCR generators. Now we focus on the case $q = \pm 1$. We will follow the approach in Example 3.8 of [9] to construct matrix model for $g_{q,k}$'s. First, we fix $m \in \mathbb{N}$ and restrict our attention to $g_{q,1}, \dots, g_{q,m}$. By (2.2) we need $2m$ semi-circular variables to approximate $g_{q,k}$'s. Thus, we consider an associated weight ψ on $N = \ell_\infty^{2m}(M_2)$. Let K be the real Hilbert space consisting of self-adjoint elements of N equipped with the inner product

$$\langle x, y \rangle_K = \frac{\psi(xy) + \psi(yx)}{2}.$$

Then $\tilde{U}_t(m) = U_t(m) \otimes I$ is a one-parameter group of unitaries on K , where

$$U_t(m) = \bigoplus_{k=1}^m U_t^k.$$

By the relationship

$$\psi(xy) = \langle x, y \rangle_{\tilde{U}(m)}$$

for self-adjoint elements x, y in Example 3.8 of [9] we can determine ψ as follows.

$$\psi(x) = \sum_{1 \leq |k| \leq m} [(2 - \sigma_k)x_{11}(k) + \sigma_k x_{22}(k)]$$

for $x = [(x_{ij}(\pm 1)), \dots, (x_{ij}(\pm m))]\in \ell_\infty^{2m}(M_2)$, where

$$\sigma_k = \frac{2\mu_{|k|}^2}{\lambda_{|k|}^2 + \mu_{|k|}^2}.$$

Let

$$(2.12) \quad u_n(x) = \sqrt{\frac{4m}{n}} \sum_{k=1}^n v_k \otimes \pi_k(x),$$

where $\pi_k : N \rightarrow N^{\otimes n}$ is the homomorphism which sends N in the k -th component and $v_k \in M_{2^n}$ are self-adjoint unitaries such that

$$(2.13) \quad v_k v_j = -v_j v_k \text{ (when } q = -1) \text{ or } v_k v_j = v_j v_k \text{ (when } q = 1).$$

Note that we are using the scaling factor $4m$ since $\psi(1_N) = 4m$ in this situation. Actually, we will use the following special choice of v_k 's. Let

$$v_{k,k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and for } j < k \text{ } v_{j,k} = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}.$$

Then we set

$$(2.14) \quad v_k = v_{1,k} \otimes \cdots \otimes v_{k,k} \otimes 1 \otimes \cdots \otimes 1 \in M_{2^n}.$$

Let \mathcal{U} be a free ultrafilter on \mathbb{N} and

$$N_{\mathcal{U}} = \left(\prod_{n, \mathcal{U}} (M_{2^n} \otimes N^{\otimes n})_* \right)^*$$

with the ultraproduct state

$$\phi_{\mathcal{U}} = (\tau_n \otimes \left(\frac{\psi}{4m} \right)^{\otimes n})_{n, \mathcal{U}},$$

where τ_n is the normalized trace on M_{2^n} . Let $(\delta_k)_{1 \leq |k| \leq m}$ be the unit vectors in ℓ_{∞}^{2m} . In [9] it is shown that $u_n(\delta_k \otimes e_{12})$ (with respect to $\phi_{\mathcal{U}}$) converges in $*$ -distribution to $s(\tilde{e}_k)$ (with respect to ϕ) as n goes infinity, and if we consider the map

$$\Phi : L_p(\Gamma_{\pm 1}) \rightarrow L_p(N_{\mathcal{U}}) \quad (1 \leq p < \infty)$$

defined by

$$\Phi(D_{\pm 1}^{\frac{1}{2p}} P(s(\tilde{e})_{\pm 1}, \dots, s(\tilde{e})_{\pm m}) D_{\pm 1}^{\frac{1}{2p}}) = D_{\phi_{\mathcal{U}}}^{\frac{1}{2p}} P(u_n(\delta_{\pm 1} \otimes e_{12}), \dots, u_n(\delta_{\pm m} \otimes e_{12})) D_{\phi_{\mathcal{U}}}^{\frac{1}{2p}}$$

for any noncommutative polynomial P , then we know that

$$(2.15) \quad I_{\mathcal{M}} \otimes \Phi : L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1}) \rightarrow L_p(\mathcal{M} \bar{\otimes} N_{\mathcal{U}})$$

is an isometry for any von Neumann algebra \mathcal{M} .

When $q = 1$ we shall understand $s(\tilde{e}_k) D_1^{\frac{1}{2p}}$ as $\frac{d}{idt} (\exp(its(\tilde{e}_k)) D_1^{\frac{1}{2p}})|_{t=0}$.

2.4. Notations. In the following we will frequently use the index sets

$$\mathcal{I}^d = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{N}^d : i_1 \neq \dots \neq i_d\}$$

and for $m \in \mathbb{N}$

$$\mathcal{I}_m^d = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{N}^d : 1 \leq i_1 \neq \dots \neq i_d \leq m\},$$

where $i_1 \neq \dots \neq i_d$ means that there is no repeating index. If we remove the restriction of repetition, then we have the index sets \mathbb{N}^d and

$$\mathbb{N}_m^d = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{N}^d : 1 \leq i_1, \dots, i_d \leq m\},$$

respectively

For $\underline{i} \in \mathcal{I}_m^d$ or \mathbb{N}_m^d we will use the notation

$$g_{q, \underline{i}} := g_{q, i_1} \cdots g_{q, i_d},$$

$$g_{p, q, \underline{i}} := D_q^{\frac{1}{2p}} g_{q, i_1} \cdots g_{q, i_d} D_q^{\frac{1}{2p}}$$

and

$$e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_d}.$$

Similarly, for a sequence of real numbers $\lambda = (\lambda_k)_{k \geq 1}$ we will write

$$\lambda_{\underline{i}} = \lambda_{i_1} \cdots \lambda_{i_d}.$$

We use the symbol $a \lesssim b$ if there is a $C > 0$ such that $a \leq Cb$ and $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. Similarly, $a \lesssim_{c_d} b$ (resp. $a \lesssim_{c_{q,d}} b$) if there is a constants $C > 0$ depending only on d (resp. only on q and d) such that $a \leq Cb$. We write $a \sim_{c_d} b$ if $a \lesssim_{c_d} b$ and $b \lesssim_{c_d} a$, and the meaning of $a \sim_{c_{q,d}} b$ is similar.

From now on we fix two sequences of strictly positive reals $\lambda = (\lambda_k)_{k \geq 1}$ and $\mu = (\mu_k)_{k \geq 1}$, the number of degree $d \in \mathbb{N}$ and a von Neumann algebra \mathcal{M} . \mathcal{M} is equipped with a distinguished normal semifinite faithful weight φ which induces the trace functional $\text{tr}_{\mathcal{M}}$ on $L_1(\mathcal{M})$. We denote the unit of \mathcal{M} by $1_{\mathcal{M}}$ whilst $I_{\mathcal{M}}$ implies the identity map on \mathcal{M} .

3. THE CASE $-1 < q < 1$

3.1. Free case: Building a model. First, we consider the free case ($q = 0$) and obtain an interpolation result as a corollary. We will simply write g_k , Γ and ϕ instead of $g_{0,k}$, Γ_0 and ϕ_0 , respectively. For $\underline{i} \in \mathcal{I}_m^d$ or \mathbb{N}_m^d we will use the notations

$$g_{\underline{i}} := g_{i_1} \cdots g_{i_d} \text{ and } g_{p,\underline{i}} := D^{\frac{1}{2p}} g_{i_1} \cdots g_{i_d} D^{\frac{1}{2p}}.$$

For $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d} \subseteq L_p(\mathcal{M})$ and $0 \leq k \leq d$ we denote by

$$\mathcal{RC}_p^{d,k}(\lambda, \mu; x) = \left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes f^{p,i_1} \otimes \cdots \otimes f^{p,i_k} \otimes f_{p,i_{k+1}} \otimes \cdots \otimes f_{p,i_d} \right\|_{L_p(\mathcal{M}; E_p^{d,k})},$$

where

$$f^{p,i} = \lambda_i^{\frac{1}{p'}} \mu_i^{\frac{1}{p}} e_{i1} \in C_p, \quad f_{p,i} = \lambda_i^{\frac{1}{p}} \mu_i^{\frac{1}{p'}} e_{1i} \in R_p$$

and

$$E_p^{d,k} = C_p^{\otimes k} \otimes_h R_p^{\otimes d-k} \subseteq S_p(\ell_2^{\otimes d}).$$

We define the corresponding \mathcal{K} - and \mathcal{J} -functional spaces $\mathcal{K}_p^d(\lambda, \mu)$ ($1 \leq p \leq 2$) and $\mathcal{J}_p^d(\lambda, \mu)$ ($2 \leq p \leq \infty$) as the closures of finite tuples in $L_p(\mathcal{M})$ indexed by \mathcal{I}^d with respect to the following norms.

$$\|x\|_{\mathcal{K}_p^d(\lambda, \mu)} = \inf \left\{ \mathcal{RC}_p^{d,0}(\lambda, \mu; x^0) + \mathcal{RC}_p^{d,1}(\lambda, \mu; x^1) + \cdots + \mathcal{RC}_p^{d,d}(\lambda, \mu; x^d) \right\},$$

where the infimum runs over all possible $x_{\underline{i}} = x_{\underline{i}}^0 + \cdots + x_{\underline{i}}^d$ and $x^k = (x_{\underline{i}}^k)_{\underline{i} \in \mathcal{I}_m^d}$ for $0 \leq k \leq d$, and

$$\|x\|_{\mathcal{J}_p^d(\lambda, \mu)} = \max_{0 \leq k \leq d} \left\{ \mathcal{RC}_p^{d,0}(\lambda, \mu; x), \mathcal{RC}_p^{d,1}(\lambda, \mu; x)_p, \dots, \mathcal{RC}_p^{d,d}(\lambda, \mu; x) \right\}.$$

Note that $\mathcal{J}_{p'}^d(\lambda, \mu) = (\mathcal{K}_p^d(\lambda, \mu))^*$ with the duality bracket

$$\langle (x_{\underline{i}})_{\underline{i} \in \mathcal{I}^d}, (z_{\underline{i}})_{\underline{i} \in \mathcal{I}^d} \rangle = \sum_{\underline{i} \in \mathcal{I}^d} \lambda_{\underline{i}} \mu_{\underline{i}} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{i}}).$$

Theorem 3.1. *Let $m \in \mathbb{N}$ and $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d} \subseteq L_p(\mathcal{M})$. Then we have the following equivalence.*

$$\left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes g_{p,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma)} \sim_{c_d} \begin{cases} \|x\|_{\mathcal{K}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2 \\ \|x\|_{\mathcal{J}_p^d(\lambda, \mu)} & \text{for } 2 \leq p \leq \infty. \end{cases}$$

Moreover, \mathcal{G}_p^d , the closed subspace of $L_p(\Gamma)$ spanned by

$$\{g_{p,\underline{i}} : \underline{i} \in \mathcal{I}^d\}$$

is completely complemented with the constant depending only on d .

Proof. We consider the following maps.

$$u_p : \begin{cases} \mathcal{K}_p^d(\lambda, \mu) & (1 \leq p \leq 2) \\ \mathcal{J}_p^d(\lambda, \mu) & (2 \leq p \leq \infty) \end{cases} \rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma),$$

$$(x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d} \mapsto \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes g_{p,\underline{i}}.$$

What we need to do is to show that u_p 's are isomorphisms with bounded constants. By a usual density argument it is enough to check that u_p 's are isomorphisms when they are restricted to the span of $(x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d}$ for a fixed $m \in \mathbb{N}$ with constants independent of m .

First, we consider the case $2 \leq p \leq \infty$ and will use an induction on d . When $d = 1$ we are done by Theorem 3.1 of [27] (or Theorem 5.1 of [10]). Suppose we have the result for d and consider the case for $d + 1$.

We will simply write $(i, \underline{i}) \in \mathcal{I}_m^{d+1}$ when we have

$$\underline{i} \in \mathcal{I}_m^d \text{ and } 1 \leq i \leq m \text{ satisfying } i \neq i_k, 1 \leq k \leq d.$$

In this case we will use the notation $x_{i, \underline{i}}$ and $g_{i, \underline{i}}$ instead of $x_{\underline{i}'}$ and $g_{\underline{i}'}$, respectively, where

$$\underline{i}' = (i, i_1, \dots, i_d) \in \mathcal{I}_m^{d+1}.$$

For $x = (x_{i, \underline{i}})_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} \subseteq L_p(\mathcal{M})$ we consider

$$\begin{aligned} \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} x_{i, \underline{i}} \otimes D^{\frac{1}{2p}} g_{i, \underline{i}} D^{\frac{1}{2p}} &= \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} (1_{\mathcal{M}} \otimes D^{\frac{1}{2p}} g_i)(x_{i, \underline{i}} \otimes g_{\underline{i}} D^{\frac{1}{2p}}) \\ &= \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} \alpha_i \beta_{i, \underline{i}}. \end{aligned}$$

By applying Theorem B of [10] we have

$$\begin{aligned} &\left\| \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} \alpha_i \beta_{i, \underline{i}} \right\|_p \\ &\sim \left\| \left(\sum_{(i, \underline{i}), (j, \underline{j}) \in \mathcal{I}_m^{d+1}} \beta_{i, \underline{i}}^* \mathbb{E}(\alpha_{\underline{i}}^* \alpha_j) \beta_{j, \underline{j}} \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\sum_{(i, \underline{i}), (j, \underline{j}) \in \mathcal{I}_m^{d+1}} \alpha_i \mathbb{E}(\beta_{i, \underline{i}} \beta_{j, \underline{j}}^*) \alpha_{\underline{j}}^* \right)^{\frac{1}{2}} \right\|_p \\ &= A + B, \end{aligned}$$

where $\mathbb{E} : L_p(\mathcal{M} \bar{\otimes} \Gamma) \rightarrow L_p(\mathcal{M})$ is the projection induced from the conditional expectation $I_{\mathcal{M}} \otimes \phi$ (see section 2 of [11]).

Since we have by (2.10)

$$\begin{aligned} \mathbb{E}(\alpha_{\underline{i}}^* \alpha_j) &= 1_{\mathcal{M}} \otimes \phi(g_i^* D^{\frac{1}{p}} g_j) \\ &= \lambda_i^{-\frac{2}{p}} \mu_j^{\frac{2}{p}} 1_{\mathcal{M}} \otimes \phi(g_i^* g_j) D^{\frac{1}{p}} \\ &= \delta_{ij} \lambda_i^{\frac{2}{p'}} \mu_j^{\frac{2}{p}} 1_{\mathcal{M}} \otimes D^{\frac{1}{p}} \end{aligned}$$

we get the following by the induction hypothesis.

$$\begin{aligned} A &= \left\| \left(\sum_{i=1}^n \lambda_i^{\frac{2}{p'}} \mu_i^{\frac{2}{p}} \left(\sum_{(i, \underline{i}), (i, \underline{j}) \in \mathcal{I}_m^{d+1}} x_{i, \underline{i}}^* x_{i, \underline{j}} \otimes g_{p, \underline{i}}^* g_{p, \underline{j}} \right) \right)^{\frac{1}{2}} \right\|_p \\ &= \left\| \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} x_{i, \underline{i}} \otimes g_{p, \underline{i}} \otimes f^{p, i} \right\|_p \\ &\sim_{c_d} \max \left\{ \left\| \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} x_{i, \underline{i}} \otimes f^{p, i_1} \otimes \dots \otimes f^{p, i_d} \otimes f^{p, i} \right\|_p, \right. \\ &\quad \left\| \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} x_{i, \underline{i}} \otimes f_{p, i_1} \otimes f^{p, i_2} \otimes \dots \otimes f^{p, i_d} \otimes f^{p, i} \right\|_p, \dots, \\ &\quad \left. \left\| \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} x_{i, \underline{i}} \otimes f_{p, i_1} \otimes \dots \otimes f_{p, i_d} \otimes f^{p, i} \right\|_p \right\}. \end{aligned}$$

Similarly we have

$$\begin{aligned}\mathbb{E}(\beta_{i,\underline{i}}\beta_{j,\underline{j}}^*) &= (I_{\mathcal{M}} \otimes \phi)(x_{i,\underline{i}}^* x_{j,\underline{j}} \otimes g_{\underline{i}} D^{\frac{1}{p}} g_{\underline{j}}^*) = \lambda_{\underline{i}}^{\frac{2}{p}} \mu_{\underline{j}}^{-\frac{2}{p}} x_{i,\underline{i}}^* x_{j,\underline{j}} \otimes \phi(g_{\underline{i}} g_{\underline{j}}^*) D^{\frac{1}{p}} \\ &= \delta_{i_1 j_1} \cdots \delta_{i_d j_d} \lambda_{\underline{i}}^{\frac{2}{p}} \mu_{\underline{j}}^{-\frac{2}{p}} x_{i,\underline{i}}^* x_{j,\underline{j}} \otimes D^{\frac{1}{p}},\end{aligned}$$

so that

$$\begin{aligned}B &= \left\| \left(\sum_{\underline{i} \in \mathcal{I}_m^d} \lambda_{\underline{i}}^{\frac{2}{p}} \mu_{\underline{j}}^{-\frac{2}{p}} \left(\sum_{\substack{i,j \neq i_k \\ 0 \leq k \leq d}} x_{i,\underline{i}} x_{j,\underline{j}}^* \otimes g_{p,i} g_{p,j}^* \right) \right)^{\frac{1}{2}} \right\|_p \\ &= \left\| \sum_{(i,\underline{i}) \in \mathcal{I}_m^{d+1}} x_{i,\underline{i}} \otimes g_{p,i} \otimes f_{p,i_1} \otimes \cdots \otimes f_{p,i_d} \right\|_p \\ &\sim \max \left\{ \left\| \sum_{(i,\underline{i}) \in \mathcal{I}_m^{d+1}} x_{i,\underline{i}} \otimes f_{p,i_1} \otimes \cdots \otimes f_{p,i_d} \otimes f_{p,i} \right\|_p, \right. \\ &\quad \left. \left\| \sum_{(i,\underline{i}) \in \mathcal{I}_m^{d+1}} x_{i,\underline{i}} \otimes f_{p,i_1} \otimes \cdots \otimes f_{p,i_d} \otimes f^{p,i} \right\|_p \right\}.\end{aligned}$$

Consequently, we get the result for $d+1$.

Now we apply a duality argument for the case $1 \leq p \leq 2$. By the above result we know that u_p 's ($2 \leq p \leq \infty$) are isomorphisms with bounded constants independent of m . Note that for $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d}$, $z = (z_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d}$ and $1 \leq p \leq 2$ we have by (2.10)

$$\begin{aligned}(3.1) \quad \langle u_p(x), u_{p'}(z) \rangle &= \sum_{\underline{i}, \underline{j} \in \mathcal{I}_m^d} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \text{tr}_{\Gamma}(g_{p,\underline{i}}^* g_{p',\underline{j}}) \\ &= \sum_{\underline{i}, \underline{j} \in \mathcal{I}_m^d} \lambda_{\underline{i}}^{-\frac{1}{p}} \mu_{\underline{i}}^{\frac{1}{p}} \lambda_{\underline{j}}^{-\frac{1}{p'}} \mu_{\underline{j}}^{\frac{1}{p'}} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \text{tr}_{\Gamma}(D^{\frac{1}{p}} g_{\underline{i}}^* g_{\underline{j}} D^{\frac{1}{p'}}) \\ &= \sum_{\underline{i}, \underline{j} \in \mathcal{I}_m^d} \lambda_{\underline{i}}^{-\frac{1}{p}} \mu_{\underline{i}}^{\frac{1}{p}} \lambda_{\underline{j}}^{-\frac{1}{p'}} \mu_{\underline{j}}^{\frac{1}{p'}} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \phi(g_{\underline{i}}^* g_{\underline{j}}) \\ &= \sum_{\underline{i} \in \mathcal{I}_m^d} \lambda_{\underline{i}} \mu_{\underline{i}} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{i}}) = \langle x, z \rangle.\end{aligned}$$

Moreover, for any $Y \in L_p(\mathcal{M} \bar{\otimes} \Gamma)$ we have

$$(3.2) \quad \langle u_p(x), Y \rangle = \langle u_p(x), u_{p'} u_p^*(Y) \rangle.$$

Combining (3.1) and (3.2) it is enough to show that $u_{p'} u_p^*$'s ($1 \leq p \leq 2$) are bounded with constants independent of m . It is straightforward to check that the maps

$$u_{p'} u_p^* : L_{p'}(\mathcal{M} \bar{\otimes} \Gamma) \rightarrow L_{p'}(\mathcal{M} \bar{\otimes} \Gamma)$$

are essentially the same maps for $1 \leq p \leq 2$ in the sense of interpolation theory. Thus, we are only to check the norms of $u_{\infty} u_1^*$ and $u_2 u_2^*$ by complex interpolation. Since $p=2$ case is trivial, it suffices to show that u_1 is bounded.

Note that

$$L_2^r(\Gamma) \widehat{\otimes} L_2^c(\Gamma) \rightarrow L_1(\Gamma), \quad a \otimes b \mapsto ab$$

is a complete contraction, so that for $0 \leq k \leq d$ we have

$$\begin{aligned} & \left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes D^{\frac{1}{2}} g_{\underline{i}} D^{\frac{1}{2}} \right\|_{L_1(\mathcal{M} \bar{\otimes} \Gamma)} \\ & \leq \left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes D^{\frac{1}{2}} g_{i_1} \cdots g_{i_k} \otimes g_{i_{k+1}} \cdots g_{i_d} D^{\frac{1}{2}} \right\|_{L_1(\mathcal{M}) \hat{\otimes} L_2^r(\Gamma) \hat{\otimes} L_2^s(\Gamma)}. \end{aligned}$$

Let $\underline{i}^k = (i_1, \dots, i_k) \in \mathcal{I}_m^k$ and $\underline{i}^{d-k} = (i_{k+1}, \dots, i_d) \in \mathcal{I}_m^{d-k}$. Since $\{D^{\frac{1}{2}} g_{\underline{i}^k}\}_{\underline{i}^k \in \mathcal{I}_m^k}$ and $\{g_{\underline{i}^{d-k}} D^{\frac{1}{2}}\}_{\underline{i}^{d-k} \in \mathcal{I}_m^{d-k}}$ are orthogonal families of vectors in $L_2(\Gamma)$ and

$$\text{tr}_{\Gamma}(D^{\frac{1}{2}} g_{\underline{i}^k} g_{\underline{i}^k}^* D^{\frac{1}{2}}) = \phi(g_{\underline{i}^k} g_{\underline{i}^k}^*) = (\mu_{i_1} \cdots \mu_{i_k})^2$$

and

$$\text{tr}_{\Gamma}(D^{\frac{1}{2}} g_{\underline{i}^{d-k}}^* g_{\underline{i}^{d-k}} D^{\frac{1}{2}}) = (\lambda_{i_{k+1}} \cdots \lambda_{i_d})^2,$$

we obtain complete isometries

$$\text{span}\{D^{\frac{1}{2}} g_{\underline{i}^k}\}_{\underline{i}^k \in \mathcal{I}_n^k} (\subseteq L_2^r(\Gamma)) \cong \text{span}\{f^{1,i_1} \otimes \cdots \otimes f^{1,i_k}\}_{\underline{i}^k \in \mathcal{I}_n^k} \subseteq C_1^{\otimes k}$$

and

$$\text{span}\{g_{\underline{i}^{d-k}} D^{\frac{1}{2}}\}_{\underline{i}^{d-k} \in \mathcal{I}_n^{d-k}} (\subseteq L_2^c(\Gamma)) \cong \text{span}\{f_{1,i_{k+1}} \otimes \cdots \otimes f_{1,i_d}\}_{\underline{i}^{d-k} \in \mathcal{I}_n^{d-k}} \subseteq R_1^{\otimes d-k}.$$

This implies u_1 is a contraction.

Now we have isomorphisms u_p for $1 \leq p \leq \infty$ with bounded constants (depending only on d). Thus \mathcal{G}_p^d 's are completely complemented with constants depending only on d by the following projections.

$$u_p u_{p'}^* : L_p(\mathcal{M} \bar{\otimes} \Gamma) \rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma).$$

□

Remark 3.2. The proof of Theorem 3.1 follows closely the Theorem F in [10], but we have to go a step further in proving the complementation result.

Now we get an interpolation scale we need in the following section.

Corollary 3.3. $\{\mathcal{K}_p^d(\lambda, \mu) : 1 \leq p \leq 2\} \cup \{\mathcal{J}_p^d(\lambda, \mu) : 2 \leq p \leq \infty\}$ forms an interpolation scale. In particular, for $1 < p < \infty$ and $\theta = \frac{1}{p}$ we have

$$[\mathcal{J}_{\infty}^d(\lambda, \mu), \mathcal{K}_1^d(\lambda, \mu)]_{\theta} \cong \begin{cases} \mathcal{K}_p^d(\lambda, \mu) & \text{for } 1 \leq p \leq 2 \\ \mathcal{J}_p^d(\lambda, \mu) & \text{for } 2 \leq p \leq \infty \end{cases}$$

isomorphically with constants depending only on d .

If we rephrase the above in the operator space language (see [18]), then we get the following interpolation result as a corollary. Let $1 \leq p \leq \infty$ and $f_{p,i}$ and $f_{p,i}$'s are as before. For $0 \leq k \leq d$ we denote the closed subspace of $C_p^{\otimes k} \otimes_h R_p^{\otimes d-k}$ spanned by

$$\{f^{p,i_1} \otimes \cdots \otimes f^{p,i_k} \otimes f_{p,i_{k+1}} \otimes \cdots \otimes f_{p,i_d} : \underline{i} \in \mathcal{I}^d\}$$

by $RC_p^{d,k}(\lambda, \mu)$. Now we define

$$RC_p^d(\lambda, \mu) = \begin{cases} RC_p^{d,0}(\lambda, \mu) + RC_p^{d,1}(\lambda, \mu) + \cdots + RC_p^{d,d}(\lambda, \mu) & \text{for } 1 \leq p \leq 2 \\ RC_p^{d,0}(\lambda, \mu) \cap RC_p^{d,1}(\lambda, \mu) \cap \cdots \cap RC_p^{d,d}(\lambda, \mu) & \text{for } 2 \leq p \leq \infty \end{cases}.$$

Corollary 3.4. $\{RC_p^d(\lambda, \mu) : 1 \leq p \leq \infty\}$ forms an interpolation scale, i.e. for $0 < \theta < 1$ and $1 \leq p, p_0, p_1 \leq \infty$ satisfying $\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}$ we have

$$[RC_{p_1}^d(\lambda, \mu), RC_{p_2}^d(\lambda, \mu)]_\theta \cong RC_p^d(\lambda, \mu)$$

completely isomorphically with constants depending only on d .

3.2. The case of general q . For $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d} \subseteq L_p(\mathcal{M})$ we can write

$$x = \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes e_{\underline{i}}$$

as an element of $\mathcal{J}_p^d(\lambda, \mu)$. Since P_d is defined on $\mathcal{H}^{\otimes d}$ the following is well-defined.

$$(I_{L_p(\mathcal{M})} \otimes P_d)x := \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes P_d(e_{\underline{i}}).$$

It is clear that $(I_{L_p(\mathcal{M})} \otimes P_d)x$ is also an element of $\mathcal{J}_p^d(\lambda, \mu)$.

Theorem 3.5. Let $m \in \mathbb{N}$ and $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d} \subseteq L_p(\mathcal{M})$. Then we have the following equivalence.

$$\left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes g_{p,q,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_q)} \sim_{c_{d,q}} \begin{cases} \|x\|_{\mathcal{K}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2 \\ \|(I_{L_p(\mathcal{M})} \otimes P_d)x\|_{\mathcal{J}_p^d(\lambda, \mu)} & \text{for } 2 \leq p \leq \infty \end{cases}$$

Moreover, $\mathcal{G}_{p,q}^d$, the closed subspace of $L_p(\Gamma_q)$ spanned by

$$\{g_{p,q,\underline{i}} : \underline{i} \in \mathcal{I}^d\}$$

is completely complemented with the constant depending only on d and q .

Proof. We consider the map $u_{p,q}$ as before with a suitable modification. For $1 \leq p \leq 2$

$$\begin{aligned} u_{p,q} : \mathcal{K}_p^d(\lambda, \mu) &\rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma_q) \\ x &\mapsto \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes g_{p,q,\underline{i}}, \end{aligned}$$

and for $2 \leq p \leq \infty$

$$\begin{aligned} u_{p,q} : \mathcal{J}_p^d(\lambda, \mu) &\rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma_q) \\ (I_{L_p(\mathcal{M})} \otimes P_d)x &\mapsto \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes g_{p,q,\underline{i}}. \end{aligned}$$

We observe that

$$\sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes g_{p,q,\underline{i}} = (I_{L_p(\mathcal{M})} \otimes D^{\frac{1}{2p}})(I_{L_p(\mathcal{M})} \otimes W)(\xi)(I_{L_p(\mathcal{M})} \otimes D^{\frac{1}{2p}}),$$

where

$$\xi = \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes \lambda_{\underline{i}} e_{\underline{i}}.$$

Thus, for $2 \leq p \leq \infty$ we have

$$u_{p,q}(x) = (I_{L_p(\mathcal{M})} \otimes D^{\frac{1}{2p}})(I_{L_p(\mathcal{M})} \otimes W)[(I_{L_p(\mathcal{M})} \otimes P_d^{-1})(\xi)](I_{L_p(\mathcal{M})} \otimes D^{\frac{1}{2p}}).$$

Our plan is as follows. First, we will show that $u_{\infty,q}$, $u_{2,q}$ and $u_{1,q}$ are bounded with constants depending only on d and q . Then by Corollary 3.3 and complex interpolation $u_{p,q}$'s ($1 \leq p \leq \infty$) are bounded with constants depending only on d and q , since it is clear that $u_{p,q}$'s are essentially the same map for $1 \leq p \leq 2$ and $2 \leq p \leq \infty$, respectively, in the sense of interpolation theory. Then, we will apply a duality argument to show that $u_{p,q}$'s ($1 \leq p \leq \infty$) are actually isomorphisms with

constants depending only on d and q . Indeed, for $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d}$, $z = (z_{\underline{j}})_{\underline{j} \in \mathcal{I}_m^d}$ and $1 \leq p \leq 2$

$$\begin{aligned}
(3.3) \quad & \langle u_{p,q}(x), u_{p',q}(z) \rangle \\
&= \sum_{\underline{i}, \underline{j} \in \mathcal{I}_m^d} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \text{tr}_{\Gamma_q}(D^{\frac{1}{2p}} W^*(\lambda_{\underline{i}} e_{\underline{i}}) D^{\frac{1}{2p}} D^{\frac{1}{2p'}} W(P_d^{-1} \lambda_{\underline{j}} e_{\underline{j}}) D^{\frac{1}{2p'}}) \\
&= \sum_{\underline{i}, \underline{j} \in \mathcal{I}_m^d} \lambda_{\underline{i}}^{-\frac{1}{p}} \lambda_{\underline{j}}^{-\frac{1}{p'}} \mu_{\underline{i}}^{\frac{1}{p}} \mu_{\underline{j}}^{\frac{1}{p'}} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \phi_q(W^*(\lambda_{\underline{i}} e_{\underline{i}}) W(P_d^{-1} \lambda_{\underline{j}} e_{\underline{j}})) \\
&= \sum_{\underline{i} \in \mathcal{I}_m^d} \lambda_{\underline{i}} \mu_{\underline{i}} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{i}}) = \langle x, z \rangle
\end{aligned}$$

and for any $Y \in L_p(\mathcal{M} \bar{\otimes} \Gamma_q)$ and $1 \leq p \leq \infty$

$$(3.4) \quad \langle u_{p,q}(x), Y \rangle = \langle u_{p,q}(x), u_{p',q} u_{p,q}^*(Y) \rangle.$$

If we know that $u_{p,q}$'s ($1 \leq p \leq \infty$) are bounded with constants depending only on d and q , then we can conclude that $u_{p,q}$'s ($1 \leq p \leq \infty$) are isomorphisms with constants depending only on d and q using (3.3) and (3.4).

Now we consider the boundedness of $u_{\infty,q}$ following the approach of Nou ([16]). For $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d}$ we have by (2.4) and (2.11)

$$\begin{aligned}
u_{\infty,q}(x) &= \sum_{k=0}^d I_{\mathcal{M}} \otimes [U_k(I_k \otimes S)(I_k \otimes S) R_{d,d-k}^*](\tilde{\xi}) \\
&= \sum_{k=0}^d I_{\mathcal{M}} \otimes [U_k(I_k \otimes S)(I_k \otimes J)(I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^*](\tilde{\xi}),
\end{aligned}$$

where $\tilde{\xi} = (I_{\mathcal{M}} \otimes P_d^{-1})(\xi)$ and I_k is the formal identity on \mathcal{H}^k .

Since $\|U_k(I_k \otimes S)\|_{cb} \leq C_q$ and $\|I_k \otimes J\|_{cb} \leq 1$ we have

$$\begin{aligned}
& \|u_{\infty,q}(x)\|_{\mathcal{M} \otimes_{\min} \Gamma_q} \\
& \leq C_q \sum_{k=0}^d \left\| I_{\mathcal{M}} \otimes (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^*(\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min} (H_c^{\otimes k} \otimes_h H_r^{\otimes d-k})}.
\end{aligned}$$

By (2.9) and (2.8) we have

$$\begin{aligned}
& \left\| I_{\mathcal{M}} \otimes (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^*(\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min} (H_c^{\otimes k} \otimes_h H_r^{\otimes d-k})} \\
&= \left\| I_{\mathcal{M}} \otimes [(P_k^{\frac{1}{2}} \otimes P_{d-k}^{\frac{1}{2}})(I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^*](\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min} (\mathcal{H}_c^{\otimes k} \otimes_h \mathcal{H}_r^{\otimes d-k})} \\
&\sim_{c_{q,d}} \left\| I_{\mathcal{M}} \otimes [(P_k \otimes P_{d-k})(I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^*](\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min} \mathcal{H}_c^{\otimes k} \otimes_h \mathcal{H}_r^{\otimes d-k}}.
\end{aligned}$$

Since P_{d-k} and $(A^{-\frac{1}{2}})^{\otimes d-k}$ commute we have by (2.5) that

$$\begin{aligned}
(P_k \otimes P_{d-k})(I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^* &= (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k})(P_k \otimes P_{d-k}) R_{d,d-k}^* \\
&= (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) P_d.
\end{aligned}$$

Thus, by (2.3) we have

$$\begin{aligned}
& \left\| I_{\mathcal{M}} \otimes (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^*(\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min}(H_c^{\otimes k} \otimes_h H_r^{\otimes d-k})} \\
& \sim_{c_{q,d}} \left\| [I_{\mathcal{M}} \otimes (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k})](\xi) \right\|_{\mathcal{M} \otimes_{\min}(\mathcal{H}_c^{\otimes k} \otimes_h \mathcal{H}_r^{\otimes d-k})} \\
& = \left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes \lambda_{\underline{i}^k} e_{\underline{i}^k} \otimes \mu_{\underline{i}^{d-k}} e_{\underline{i}^{d-k}} \right\|_{\mathcal{M} \otimes_{\min}(\mathcal{H}_c^{\otimes k} \otimes_h \mathcal{H}_r^{\otimes d-k})} \\
& = \mathcal{RC}_{\infty}^{d,k}(\lambda, \mu; x).
\end{aligned}$$

Consequently, we get

$$\|u_{\infty,q}(x)\|_{\mathcal{M} \otimes_{\min} \Gamma_q} \lesssim_{c_{q,d}} \|x\|_{\mathcal{J}_{\infty}^d(\lambda, \mu)}.$$

When $p = 2$ the calculation is straightforward. We can easily check that

$$\begin{aligned}
\|u_{2,q}(x)\|_{L_2(\mathcal{M} \otimes \Gamma_q)} &= \left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes \lambda_{\underline{i}^{\frac{1}{2}}} \mu_{\underline{i}^{\frac{1}{2}}} e_{\underline{i}} \right\|_{L_2(\mathcal{M}) \otimes_2 H^{\otimes d}} \\
&= \left\| (I_{L_2(\mathcal{M})} \otimes P_d^{\frac{1}{2}}) \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes \lambda_{\underline{i}^{\frac{1}{2}}} \mu_{\underline{i}^{\frac{1}{2}}} e_{\underline{i}} \right\|_{L_2(\mathcal{M}) \otimes_2 \mathcal{H}^{\otimes d}} \\
&= \left\| (I_{L_2(\mathcal{M})} \otimes P_d^{\frac{1}{2}}) x \right\|_{\mathcal{J}_2^d(\lambda, \mu)}.
\end{aligned}$$

Since

$$\left\| (I_{L_2(\mathcal{M})} \otimes P_d^{\frac{1}{2}}) x \right\|_{\mathcal{J}_2^d(\lambda, \mu)} \sim_{c_{q,d}} \left\| (I_{L_2(\mathcal{M})} \otimes P_d) x \right\|_{\mathcal{J}_2^d(\lambda, \mu)} \sim_{c_{q,d}} \|x\|_{\mathcal{K}_2^d(\lambda, \mu)}$$

by (2.7) and (2.8) we get our desired estimate.

The boundedness of $u_{1,q}$ can be checked by the identical argument as in the proof of Theorem 3.1, so we are done here. \square

4. THE CASE OF CAR AND CCR GENERATORS

We will simply write a_k , $a_{p,k}$ and \mathcal{A}_p^d instead of $g_{\pm 1,k}$, $g_{p,\pm 1,k}$ and $\mathcal{G}_{p,\pm 1}^d$, respectively. For any $\underline{i} \in \mathbb{N}_m^d$ we define $a_{p,\underline{i}}$ similarly. We start this section with the operator space structure of \mathcal{A}_p^1 .

Theorem 4.1. *Let $m \in \mathbb{N}$ and $x = (x_k)_{1 \leq k \leq m} \subseteq L_p(\mathcal{M})$. Then we have*

$$\left\| \sum_{1 \leq k \leq m} x_k \otimes a_{p,k} \right\|_{L_p(\mathcal{M} \otimes \Gamma_{\pm 1})} \sim_{c_p} \begin{cases} \|x\|_{\mathcal{K}_p^1(\lambda, \mu)} & \text{for } 1 \leq p \leq 2 \\ \|x\|_{\mathcal{J}_p^1(\lambda, \mu)} & \text{for } 2 \leq p < \infty. \end{cases}$$

Moreover, $L_p(\mathcal{M}; \mathcal{A}_p)$ is complemented in $L_p(\mathcal{M} \otimes \Gamma_{\pm 1})$.

In particular, we have $\mathcal{A}_p^1 \cong RC_p(\lambda, \mu)$ ($1 \leq p < \infty$) completely isomorphically, and \mathcal{A}_p^1 ($1 < p < \infty$) is completely complemented in $L_p(\Gamma_{\pm 1})$. All constants here depend only on p .

Proof. When $2 \leq p < \infty$ we apply noncommutative Burkholder inequality (also see Theorem 4.1 of [26]). For $1 < p < 2$ we follow the proof of Theorem E in [10] to obtain an upper bound. The lower bound follows by the duality and the previous result.

For the case $p = 1$ we can get the result with a slight modification of the proof of Theorem 7.1 in [9]. Let ψ , u_n and N be as in section 2.3. Let $N_n = M_{2^n} \otimes N^{\otimes n}$

and D_ψ , D_n be the densities of ψ and $\tau_n \otimes (\frac{\psi}{4m})^{\otimes n}$, respectively. Then by (2.2) and (2.15) we have

$$\begin{aligned} & \left\| \sum_{k=1}^m x_k \otimes a_{1,k} \right\|_{L_1(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \\ &= \lim_n \left\| \sum_{k=1}^m \frac{\sqrt{\lambda_k^2 + \mu_k^2}}{2} x_k \otimes D_n^{\frac{1}{2}} u_n((\delta_k + i\delta_{-k}) \otimes e_{12}) D_n^{\frac{1}{2}} \right\|_{L_1(\mathcal{M} \bar{\otimes} N_n)}. \end{aligned}$$

The only difference now is that we are using $(\delta_k + i\delta_{-k}) \otimes e_{12}$ instead of $\delta_k \otimes e_{12}$ in (7.3) of [9]. Since

$$\begin{aligned} & \text{tr}_N([D_\psi^{\frac{1}{2}}((\delta_l - i\delta_{-l}) \otimes e_{21})]^* D_\psi^{\frac{1}{2}}((\delta_k + i\delta_{-k}) \otimes e_{12})) \\ &= \text{tr}_N((\delta_l - i\delta_{-l}) \otimes e_{21}) D_\psi((\delta_k + i\delta_{-k}) \otimes e_{12}) \\ &= 2\delta_{kl}(2 - \sigma_k) = \delta_{kl} \frac{4\lambda_k^2}{\lambda_k^2 + \mu_k^2}. \end{aligned}$$

by the same argument we get

$$\begin{aligned} & \left\| \sum_{k=1}^m x_k \otimes a_{1,k} \right\|_{L_1(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \\ & \sim \inf_{x_k = c_k + d_k} \left\{ \left\| \left(\sum_{k=1}^m \lambda_k^2 c_k^* c_k \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{M})} + \left\| \left(\sum_{k=1}^m \mu_k^2 d_k d_k^* \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{M})} \right\} \\ &= \|(x_k)_{k=1}^m\|_{\mathcal{K}_1^1(\lambda, \mu)}. \end{aligned}$$

□

For $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$ and $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_d) \in \{c, r\}^d$ we denote

$$\mathcal{RC}_p^{\mathbf{r}}(\lambda, \mu; x) = \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes h_{p, \mathbf{r}_1, i_1} \otimes \dots \otimes h_{p, \mathbf{r}_d, i_d} \right\|_{L_p(\mathcal{M}; F_p^{\mathbf{r}})},$$

where

$$\begin{aligned} h_{p, c, i} &= \lambda_i^{\frac{1}{p}} \mu_i^{\frac{1}{p}} e_{i1} \in C_p, \quad h_{p, r, i} = \lambda_i^{\frac{1}{p}} \mu_i^{\frac{1}{p}} e_{1i} \in R_p, \\ F_p^{\mathbf{r}} &= F_1 \otimes_h \dots \otimes_h F_d \subseteq S_p(\ell_2^{\otimes d}) \end{aligned}$$

and

$$F_k = \begin{cases} C_p & \text{if } \mathbf{r}_k = c \\ R_p & \text{if } \mathbf{r}_k = r. \end{cases}$$

We define the corresponding symmetric \mathcal{K} - and \mathcal{J} -functional spaces $\mathcal{SK}_p^d(\lambda, \mu)$ ($1 \leq p \leq 2$) and $\mathcal{SJ}_p^d(\lambda, \mu)$ ($2 \leq p \leq \infty$) as the closures of finite tuples in $L_p(\mathcal{M})$ indexed by \mathbb{N}^d with respect to the following norms.

$$\|x\|_{\mathcal{SK}_p^d(\lambda, \mu)} = \inf_{\mathbf{r} \in \{c, r\}^d} \mathcal{RC}_p^{\mathbf{r}}(\lambda, \mu; x^{\mathbf{r}}),$$

where the infimum runs over all possible $x_{\underline{i}} = \sum_{\mathbf{r} \in \{c, r\}^d} x_{\underline{i}}^{\mathbf{r}}$ and $x^{\mathbf{r}} = (x_{\underline{i}}^{\mathbf{r}})_{\underline{i} \in \mathbb{N}_m^d}$ for $\mathbf{r} \in \{c, r\}^d$, and

$$\|x\|_{\mathcal{SJ}_p^d(\lambda, \mu)} = \max_{\mathbf{r} \in \{c, r\}^d} \mathcal{RC}_p^{\mathbf{r}}(\lambda, \mu; x).$$

The following is the tensor product extension of Theorem 4.1.

Corollary 4.2. For $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$ and $1 \leq p < \infty$ we have the following equivalence.

$$\left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,i_1} \otimes \cdots \otimes a_{p,i_d} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1}^{\otimes d})} \sim_{c_{p,d}} \begin{cases} \|x\|_{\mathcal{SK}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2 \\ \|x\|_{\mathcal{SJ}_p^d(\lambda, \mu)} & \text{for } 2 \leq p < \infty. \end{cases}$$

Proof. We assume that $1 \leq p \leq 2$ and $d = 2$. The proof for the other cases are the same. By Theorem 4.1 we have

$$\begin{aligned} & \left\| \sum_{i,j=1}^m x_{ij} \otimes a_{p,i} \otimes a_{p,j} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1} \bar{\otimes} \Gamma_{\pm 1})} \\ & \sim_{c_p} \inf_{x_{ij}=c_{ij}+d_{ij}} \left\{ \left\| \left(\sum_{j=1}^m \left(\sum_{i=1}^m c_{ij} \otimes a_{p,i} \right) \otimes \lambda_j^{\frac{1}{p}} \mu_j^{\frac{1}{p}} e_{j1} \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1} \bar{\otimes} B(\ell_2))} \right. \\ & \quad \left. + \left\| \left(\sum_{j=1}^m \left(\sum_{i=1}^m d_{ij} \otimes a_{p,i} \right) \otimes \lambda_j^{\frac{1}{p}} \mu_j^{\frac{1}{p}} e_{1j} \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1} \bar{\otimes} B(\ell_2))} \right\}. \end{aligned}$$

If we apply Theorem 4.1 for $\mathcal{M} \bar{\otimes} B(\ell_2)$, then we get the desired result. \square

4.1. Polynomials of degree 2. In this section we first focus on the case $d = 2$ to get the idea for polynomials of higher degree.

Theorem 4.3. Let $x = (x_{ij})_{i,j=1}^m \subseteq L_p(\mathcal{M})$ satisfying

$$(4.1) \quad x_{ij} = -x_{ji} \text{ for } q = -1$$

and

$$x_{ij} = x_{ji} \text{ for } q = 1$$

for all $1 \leq i, j \leq m$. Then we have the following equivalence.

$$\left\| \sum_{i,j=1}^m x_{ij} \otimes D_{\pm 1}^{\frac{1}{2p}} a_i a_j D_{\pm 1}^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \sim_{c_p} \begin{cases} \|x\|_{\mathcal{SK}_p^2(\lambda, \mu)} & \text{for } 1 \leq p \leq 2 \\ \|x\|_{\mathcal{SJ}_p^2(\lambda, \mu)} & \text{for } 2 \leq p < \infty. \end{cases}$$

Proof. Let u_n and v_k be as in (2.12) and (2.14), respectively. By (2.15) we have

$$\begin{aligned} & \left\| \sum_{i,j=1}^m x_{ij} \otimes D_{\pm 1}^{\frac{1}{2p}} a_i a_j D_{\pm 1}^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \\ & = \lim_n \left\| \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}} (u_n(b_i) u_n(b_j)) D_n^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} N_n)}, \end{aligned}$$

where $b_k = \frac{\sqrt{\lambda_k^2 + \mu_k^2}}{2} (\delta_k + i\delta_{-k}) \otimes e_{12}$.

Note that we have

$$u_n(b_i) u_n(b_j) = \frac{4m}{n} \sum_{k,l=1}^n v_k v_l \otimes \pi_k(b_i) \pi_l(b_j)$$

and $\psi_n(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j)) = 0$ for $k \neq l$. Thus, we have

$$\begin{aligned} & \left\| \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}}(u_n(b_i) u_n(b_j)) D_n^{\frac{1}{2p}} \right\|_p \\ &= \frac{4m}{n} \left\| \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}} \left(\sum_{k,l=1}^n v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p. \end{aligned}$$

We set

$$Q_n = \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq k,l \leq n}} x_{ij} \otimes D_n^{\frac{1}{2p}} \left(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}}.$$

First, we show the lower bound by an averaging technique. Let us fix a subset $A \subseteq \{1, \dots, n\}$. We introduce

$$v_k(\varepsilon) = \begin{cases} \varepsilon_1 v_k & \text{for } k \in A \\ \varepsilon_2 v_k & \text{for } k \notin A \end{cases},$$

where ε_1 and ε_2 are two independent Bernoulli variables with $\text{Prob}(\varepsilon_i = \pm 1) = \frac{1}{2}$. Then

$$\begin{aligned} & \int_0^1 \int_0^1 \varepsilon_1(t_1) \varepsilon_2(t_2) \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq k,l \leq n}} x_{ij} \otimes v_k(\varepsilon) v_l(\varepsilon) \otimes \pi_k(b_i) \pi_l(b_j) dt_1 dt_2 \\ &= \sum_{\substack{1 \leq i,j \leq m \\ k \in A, l \notin A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) + \sum_{\substack{1 \leq i,j \leq m \\ k \notin A, l \in A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \\ &= 2 \sum_{\substack{1 \leq i,j \leq m \\ k \in A, l \notin A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \end{aligned}$$

since (when $q = -1$)

$$\begin{aligned} \sum_{\substack{1 \leq i,j \leq m \\ k \in A, l \notin A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) &= \sum_{\substack{1 \leq i,j \leq m \\ k \notin A, l \in A}} x_{ij} \otimes v_l v_k \otimes \pi_l(b_i) \pi_k(b_j) \\ &= - \sum_{\substack{1 \leq i,j \leq m \\ k \notin A, l \in A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_j) \pi_l(b_i) \\ &= \sum_{\substack{1 \leq i,j \leq m \\ k \notin A, l \in A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j). \end{aligned}$$

The case $q = 1$ is the same.

Clearly the tuple $((v_k(\varepsilon))_{k \in A}, (v_k(\varepsilon))_{k \notin A})$ has the same $*$ -distribution as the tuple $((v_k)_{k \in A}, (v_k)_{k \notin A})$. Thus, by combining the above observations we get

$$(4.2) \quad \left\| \sum_{\substack{1 \leq i,j \leq m \\ k \in A, l \notin A}} x_{ij} \otimes D_n^{\frac{1}{2p}} \left(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p \leq \frac{1}{2} \|Q_n\|_p.$$

Now we assume that n is even and consider a specific choice of A , namely

$$A = \{1, \dots, \frac{n}{2}\}.$$

Then $k \in A$, $l \notin A$ means $1 \leq k \leq \frac{n}{2} < l \leq n$. In this case we decompose v_k and v_l as follows.

$$v_k = w_k \otimes 1, \quad v_l = u \otimes w_l,$$

where w_k, w_l and $u \in M_{2^{n/2}}$. Moreover, $\pi_k(b_i)\pi_l(b_j)$ can be understood as $\pi_k(b_i) \otimes \pi_l(b_j)$ since $\pi_k(b_i) \in N^{\otimes \frac{n}{2}} \otimes 1$ and $\pi_l(b_j) \in 1 \otimes N^{\otimes \frac{n}{2}}$. Since $D_n = I \otimes D_{\psi/4m}^{\otimes n}$ we can write $D_n = D_{n/2} \otimes D_{n/2}$. Now we have

$$\begin{aligned} & \sum_{i,j=1}^m \sum_{1 \leq k \leq \frac{n}{2} < l \leq n} x_{ij} \otimes D_n^{\frac{1}{2p}} \left[v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right] D_n^{\frac{1}{2p}} \\ &= \sum_{i,j=1}^m \sum_{1 \leq k \leq \frac{n}{2} < l \leq n} x_{ij} \otimes D_n^{\frac{1}{2p}} \left[(w_k \otimes 1)(u \otimes w_l) \otimes \pi_k(b_i) \pi_l(b_j) \right] D_n^{\frac{1}{2p}} \\ &= \sum_{i,j=1}^m x_{ij} \otimes D_{n/2}^{\frac{1}{2p}} \left[\sum_{k=1}^{n/2} w_k u \otimes \pi_k(b_i) \right] D_{n/2}^{\frac{1}{2p}} \otimes D_{n/2}^{\frac{1}{2p}} \left[\sum_{l=n/2+1}^n w_l \otimes \pi_l(b_j) \right] D_{n/2}^{\frac{1}{2p}}. \end{aligned}$$

Since w_k and u commute $(w_k u)_{1 \leq k \leq \frac{n}{2}}$ and $(w_l)_{\frac{n}{2} < l \leq n}$ satisfy the relations in (2.13). Thus,

$$\sqrt{\frac{4m}{n/2}} \sum_{k=1}^{n/2} w_k u \otimes \pi_k(b_i) \quad \text{and} \quad \sqrt{\frac{4m}{n/2}} \sum_{l=n/2+1}^n w_l \otimes \pi_l(b_j)$$

converges in $*$ -distribution to a_i and a_j , respectively, and consequently we have

$$\begin{aligned} & \left\| \sum_{i,j=1}^m x_{ij} \otimes a_{i,p} \otimes a_{j,p} \right\|_p \\ (4.3) \quad &= \lim_n \frac{4m}{n/2} \left\| \sum_{i,j=1}^m \sum_{1 \leq k \leq \frac{n}{2} < l \leq n} x_{ij} \otimes D_n^{\frac{1}{2p}} \left(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p \\ &\leq 2 \lim_n \frac{4m}{n} \|Q_n\|_p = 2 \left\| \sum_{i,j=1}^m x_{ij} \otimes D_{\pm 1}^{\frac{1}{2p}} a_i a_j D_{\pm 1}^{\frac{1}{2p}} \right\|_p. \end{aligned}$$

The conclusion follows by Corollary 4.2.

For the upper bound we consider the discrete probability space Ω of

$$A \subseteq \{1, \dots, n\} \quad \text{with} \quad |A| = \frac{n}{2}.$$

Then for fixed $1 \leq k \neq l \leq n$ we have

$$\begin{aligned} \text{Prob}(k \in A, l \notin A) &= \frac{\#\{A \subseteq \{1, \dots, n\} : |A| = \frac{n}{2}, k \in A, l \notin A\}}{\#\{A \subseteq \{1, \dots, n\} : |A| = \frac{n}{2}\}} \\ &= \frac{\binom{n-2}{n/2-1}}{\binom{n}{n/2}} = \frac{n}{4(n-1)}. \end{aligned}$$

Thus, by the symmetry we have

$$\begin{aligned}
\|Q_n\|_p &= \frac{4(n-1)}{n} \left\| \sum_{k,l=1}^n \mathbb{E}_A(1_{k \in A, l \notin A}) \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}}(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j)) D_n^{\frac{1}{2p}} \right\|_p \\
&\leq 4 \mathbb{E}_A \left\| \sum_{k \in A, l \notin A} \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}}(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j)) D_n^{\frac{1}{2p}} \right\|_p \\
&= 4 \left\| \sum_{1 \leq k \leq n/2 < l \leq n} \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}}(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j)) D_n^{\frac{1}{2p}} \right\|_p,
\end{aligned}$$

where \mathbb{E}_A implies the expectation with respect to Ω . Consequently, by (4.3) we get

$$\left\| \sum_{i,j=1}^m x_{ij} \otimes D_{\pm 1}^{\frac{1}{2p}} a_i a_j D_{\pm 1}^{\frac{1}{2p}} \right\|_p \leq 8 \left\| \sum_{i,j=1}^m x_{ij} \otimes a_{p,i} \otimes a_{p,j} \right\|_p,$$

and the conclusion follows by Corollary 4.2 again. \square

4.2. Polynomials of higher degree. Now we consider the general case $d \geq 2$. For a permutation $\rho \in S_d$ and $\underline{i} \in \mathbb{N}_m^d$ we denote $(i_{\rho(1)}, \dots, i_{\rho(d)}) \in \mathbb{N}_m^d$ by $\underline{i}(\rho)$.

Theorem 4.4. *Let $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$ satisfying*

$$(4.4) \quad x_{\underline{i}} = \text{sgn}(\rho) x_{\underline{i}(\rho)} \text{ for } q = -1$$

$$x_{\underline{i}} = x_{\underline{i}(\rho)} \text{ for } q = 1$$

for any $\rho \in S_d$ and $\underline{i} \in \mathbb{N}_m^d$. Then we have the following equivalence.

$$\left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \sim_{c_{p,d}} \begin{cases} \|x\|_{\mathcal{SK}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2 \\ \|x\|_{\mathcal{SJ}_p^d(\lambda, \mu)} & \text{for } 2 \leq p < \infty. \end{cases}$$

Proof. As in the proof of Theorem 4.3 we have

$$\begin{aligned}
\left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} &= \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes D_{\pm 1}^{\frac{1}{2p}} a_{\underline{i}} D_{\pm 1}^{\frac{1}{2p}} \right\|_p \\
&= \lim_n \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes D_n^{\frac{1}{2p}}(u_n(b_{i_1}) \cdots u_n(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} N_n)} \\
&= \lim_n \left(\frac{4m}{n} \right)^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \sum_{\underline{j} \in \mathbb{N}_n^d} v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p,
\end{aligned}$$

and we set

$$Q_n = \sum_{\underline{i} \in \mathbb{N}_m^d, \underline{j} \in \mathbb{N}_n^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}}$$

First, we consider the lower bound. Let (A_1, \dots, A_d) be a partition of $\{1, \dots, n\}$. For $1 \leq k \leq d$ we introduce

$$v_j(\varepsilon) = \varepsilon_k v_j \text{ for all } j \in A_k,$$

where $(\varepsilon_k)_{k=1}^d$ is a collection of independent Bernoulli variables with $\text{Prob}(\varepsilon_i = \pm 1) = \frac{1}{2}$. Now we observe that

$$\begin{aligned} & \int_{[0,1]^d} \varepsilon_1 \cdots \varepsilon_d \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \sum_{\underline{j} \in \mathbb{N}_n^d} v_{\underline{j}}(\varepsilon) \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \\ &= \sum_{\rho \in S_d} \sum_{j_{\rho(1)} \in A_1, \dots, j_{\rho(d)} \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \end{aligned}$$

since the integration kills all the terms where all j_k 's belong to different partitions.

On the other hand for a fixed $\rho \in S_d$ we have (when $q = -1$)

$$\begin{aligned} & \sum_{j_{\rho(1)} \in A_1, \dots, j_{\rho(d)} \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes \pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d}) \\ &= \sum_{j_1 \in A_1, \dots, j_d \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}(\rho^{-1})} \otimes \pi_{j_{\rho^{-1}(1)}}(b_{i_1}) \cdots \pi_{j_{\rho^{-1}(d)}}(b_{i_d}) \\ &= \sum_{j_1 \in A_1, \dots, j_d \in A_d} \text{sgn}(\rho) \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes \pi_{j_1}(b_{i_{\rho^{-1}(1)}}) \cdots \pi_{j_d}(b_{i_{\rho^{-1}(d)}}) \\ &= \sum_{j_1 \in A_1, \dots, j_d \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} \text{sgn}(\rho) x_{\underline{i}(\rho)} \otimes v_{\underline{j}} \otimes \pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d}) \\ &= \sum_{j_1 \in A_1, \dots, j_d \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes \pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d}). \end{aligned}$$

The case $q = 1$ is similar.

Since $(v_j(\varepsilon))_{j=1}^n$ has the same $*$ -distribution with $(v_j)_{j=1}^n$ by combining the above two observations we get

$$d! \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} \sum_{j_1 \in A_1, \dots, j_d \in A_d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p \leq \|Q_n\|_p.$$

Now we assume that n is a multiple of d and choose

$$A_k = \left\{ \frac{(k-1)n}{d} + 1, \dots, \frac{kn}{d} \right\} \text{ for } 1 \leq k \leq d.$$

As in the case $d = 2$ we can write $v_j = u^{\otimes k-1} \otimes w_{j,k} \otimes 1$ for $j \in A_k$ and $u, w_{j,k} \in M_{2^{n/d}}$. Then

$$v_{\underline{j}} = w_{j_1,1} u^{d-1} \otimes v_{j_2,2} u^{d-2} \otimes \cdots \otimes v_{j_d,d}.$$

Moreover, we have $\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d}) = \pi_{j_1}(b_{i_1}) \otimes \cdots \otimes \pi_{j_d}(b_{i_d})$ for $j_1 \in A_1, \dots, j_d \in A_d$ and $D_n = D_{n/d} \otimes \cdots \otimes D_{n/d}$, so that

$$\begin{aligned} & \sum_{\underline{i} \in \mathbb{N}_m^d} \sum_{j_1 \in A_1, \dots, j_d \in A_d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \\ &= \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \bigotimes_{k=1}^d D_{n/d}^{\frac{1}{2p}} \left[\sum_{j_k \in A_k} v_{j_k,k} u^{d-k} \otimes \pi_{j_k}(b_{i_k}) \right] D_{n/d}^{\frac{1}{2p}} \end{aligned}$$

Since $v_{j_k,k}$ and u commute $(v_{j_k,k} u^{d-k})_{j_k \in A_k}$'s satisfy the relations in (2.13)

$$\sqrt{\frac{4m}{n/d}} \sum_{j_k \in A_k} v_{j_k,k} u^{d-k} \otimes \pi_{j_k}(b_{i_k})$$

converges in $*$ -distribution to a_{i_k} by the central limit procedure. Thus, we have

$$\begin{aligned}
& \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,i_1} \otimes \cdots \otimes a_{p,i_d} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1}^{\otimes d})} \\
&= \lim_n \left(\frac{4m}{n/d} \right)^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \bigotimes_{k=1}^d D_{n/d}^{\frac{1}{2p}} \left[\sum_{j_k \in A_k} v_{j_k, k} u^{d-k} \otimes \pi_{j_k}(b_{i_k}) \right] D_{n/d}^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} N_n)} \\
&= d^{\frac{d}{2}} \lim_n \left(\frac{4m}{n} \right)^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} \sum_{j_1 \in A_1, \dots, j_d \in A_d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p \\
&\leq \frac{d^{\frac{d}{2}}}{d!} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})}.
\end{aligned}$$

Then by Corollary 4.2 we get the lower bound.

For the upper bound we consider a fixed $(j_1, \dots, j_d) \in \mathbb{N}_m^d$ and the discrete probability space Ω of partitions $\tilde{A} = (A_1, \dots, A_d)$ of $\{1, \dots, n\}$ with $|A_k| = \frac{n}{d}$. Then we have

$$\begin{aligned}
\text{Prob}(j_1 \in A_1, \dots, j_d \in A_d) &= \frac{\#\{(A'_1, \dots, A'_d) : A'_1 \cup \dots \cup A'_d = \{1, \dots, n-d\}\}}{\#\{(A_1, \dots, A_d) : A_1 \cup \dots \cup A_d = \{1, \dots, n\}\}} \\
&= \frac{\binom{n-d}{n/d-1, \dots, n/d-1}}{\binom{n}{n/d, \dots, n/d}} = \frac{n^d(n-d)!}{d^d n!}.
\end{aligned}$$

This implies

$$\begin{aligned}
& \|Q_n\|_p \\
&= \frac{d^d n!}{n^d(n-d)!} \left\| \sum_{\substack{\underline{j} \in \mathbb{N}_n^d \\ \underline{i} \in \mathbb{N}_m^d}} \mathbb{E}_{\tilde{A}}(1_{j_1 \in A_1, \dots, j_d \in A_d}) x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p \\
&\leq d^d \mathbb{E}_{\tilde{A}} \left\| \sum_{j_1 \in A_1, \dots, j_d \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p,
\end{aligned}$$

where $\mathbb{E}_{\tilde{A}}$ is the expectation with respect to Ω , and consequently

$$\begin{aligned}
& \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} = \lim_n \left(\frac{4m}{n} \right)^{\frac{d}{2}} \|Q_n\|_p \\
&\leq d^d \lim_n \left(\frac{4m}{n} \right)^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} \sum_{j_1 \in A_1, \dots, j_d \in A_d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p \\
&= d^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,i_1} \otimes \cdots \otimes a_{p,i_d} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1}^{\otimes d})}.
\end{aligned}$$

We get the desired conclusion by Corollary 4.2 again. \square

Remark 4.5. The conditions (4.1) and (4.4) imply $x_{ii} = 0$ for any i and $x_{\underline{i}} = 0$ for any $\underline{i} \in \mathbb{N}_m^d - \mathcal{I}_m^d$.

REFERENCES

1. A. Buchholz, Operator Khintchine inequality in non-commutative probability. *Math. Ann.* **319** (2001), no. 1, 1-16.
2. M. Bożejko, Completely positive maps on Coxeter groups and the ultracontractivity of the q -Ornstein-Uhlenbeck semigroup. *Quantum probability* (Gdańsk, 1997), 87-93, Banach Center Publ., 43, Polish Acad. Sci., Warsaw, 1998.
3. M. Bożejko and R. Speicher, Roland An example of a generalized Brownian motion. *Comm. Math. Phys.* **137** (1991), no. 3, 519-531.
4. M. Bożejko and R. Speicher, Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. *Math. Ann.* **300** (1994), no. 1, 97-120.
5. E. G. Effros and Z. J. Ruan, Operator spaces. London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000.
6. F. Hiai, q -deformed Araki-Woods algebras. *Operator algebras and mathematical physics* (Constanța, 2001), 169-202, Theta, Bucharest, 2003.
7. U. Haagerup and G. Pisier, Bounded linear operators between C^* -algebras. *Duke Math. J.* **71** (1993), no. 3, 889-925.
8. M. Junge, Embedding of the operator space OH and the logarithmic 'little Grothendieck inequality'. *Invent. Math.* **161** (2005), no. 2, 225-286.
9. M. Junge, Operator spaces and Araki-Woods factors: a quantum probabilistic approach. *IMRP Int. Math. Res. Pap.* 2006, Art. ID 76978, 87 pp.
10. M. Junge and J. Parcet; Q. Xu, Rosenthal type inequalities for free chaos. *Ann. Probab.* **35** (2007), no. 4, 1374-1437.
11. M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities. *Ann. Probab.* **31** (2003), 948-995.
12. H. Kosaki, Applications of the complex interpolation method to a von Neumann algebra: noncommutative L^p -spaces. *J. Funct. Anal.* **56** (1984), no. 1, 29-78.
13. F. Lust-Piquard, Inegalites de Khintchine dans C_p ($1 < p < \infty$). (French) [Khinchin inequalities in C_p ($1 < p < \infty$)] *C. R. Acad. Sci. Paris Ser. I Math.* **303** (1986), no. 7, 289-292.
14. F. Lust-Piquard and G. Pisier, Noncommutative Khintchine and Paley inequalities. *Ark. Mat.* **29** (1991), no. 2, 241-260.
15. M. Ledoux and M. Talagrand, Probability in Banach spaces, Springer Verlag, 1992.
16. A. Nou, Non injectivity of the q -deformed von Neumann algebra. *Math. Ann.* **330** (2004), no. 1, 17-38.
17. J. Parcet and G. Pisier, Non-commutative Khintchine type inequalities associated with free groups. *Indiana Univ. Math. J.* **54** (2005), no. 2, 531-556.
18. G. Pisier, Non-commutative vector valued L_p -spaces and completely p -summing maps. *Astérisque*(Soc. Math. France) **247** (1998), 1-111.
19. G. Pisier, Introduction to operator space theory. London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003.
20. G. Pisier and Q. Xu, Non-commutative L_p -spaces. In *Handbook of the geometry of the Banach spaces, Vol. 2, 1459-1517*. North-Holland, Amsterdam, 2003.
21. E. Ricard and Q. Xu, Khintchine type inequalities for reduced free products and applications. *J. Reine Angew. Math.* **599** (2006), 27-59.
22. D. Shlyakhtenko, Free quasi-free states. *Pacific J. Math.* **177** (1997) no. 2, 329-368.
23. M. Takesaki, Theory of operator algebras. I. Springer-Verlag, New York-Heidelberg, 1979.
24. M. Takesaki, Theory of operator algebras. II. *Encyclopaedia of Mathematical Sciences*, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003.
25. M. Terp, Interpolation spaces between a von Neumann algebra and its predual. *J. Operator Theory* **8** (1982), no. 2, 327-360.
26. Q. Xu, Embedding of C_q and R_q into noncommutative L_p -spaces, $1 \leq p < q \leq 2$. *Math. Ann.* **335** (2006), no. 1, 109-131.
27. Q. Xu, Operator Grothendieck inequalities for noncommutative L_p -spaces. *Duke Math. J.* **131** (2006), no. 3, 525-574.

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